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Blaschke-oscillatory equations of the form

$$f'' + A(z)f = 0^{\star}$$

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Abstract

We study the zero sequences of the non-trivial solutions of

$$f'' + A(z)f = 0, \tag{†}$$

where $A(z)$ is analytic in the unit disc. We offer several aspects illustrating the fact that it is not so uncommon for these zero sequences to be Blaschke sequences. The typical results can be divided into two categories: (1) We search for conditions on $A(z)$ under which the zero sequences of solutions of (†) are Blaschke sequences. (2) For given Blaschke sequences satisfying certain conditions, we construct an analytic function $A(z)$ (of minimal growth) such that these Blaschke sequences are the zero sequences of certain solutions of (†).

This discussion is a continuation of the recent paper [J. Heittokangas, Solutions of $f'' + A(z)f = 0$ in the unit disc having Blaschke sequences as the zeros, *Comput. Methods Funct. Theory* 5 (2005) 49–63].

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1. Introduction

In the recent paper [5], the zero distribution of the solutions of

$$f'' + A(z)f = 0, \quad (1.1)$$

where $A(z)$ is analytic in the unit disc $D = \{z: |z| < 1\}$, is studied from the following points of view:

Problem 1. Find a growth condition for $A(z)$ such that the zero sequence of any non-trivial solution of (1.1) is a Blaschke sequence.

Problem 2. If $\{z_n\}$ is a Blaschke sequence of distinct points in D , then find a function $A(z)$, analytic in D , such that (1.1) possesses a solution having zeros precisely at the points z_n .

An equation of the form (1.1), such that the zero sequence of any non-trivial solution of (1.1) is a Blaschke sequence, is called *Blaschke-oscillatory* in [5]. This new concept generalizes the classical concepts of disconjugate and non-oscillatory equations. An equation of the form (1.1) is called *disconjugate* (respectively *non-oscillatory*), if every non-trivial solution of (1.1) has at most one zero (respectively finitely many zeros) in D . The paper [5] answers to several aspects of the more general problem of characterizing all Blaschke-oscillatory equations of the form (1.1), and the present paper continues this discussion.

The next result is a simple consequence of [11, Theorem 5]. See [5, Section 3] for a further discussion on Problem 1.

Theorem A. *If $A(z)$ is analytic in D satisfying*

$$\int_D |A(z)|^{\frac{1}{2}} d\sigma_z < \infty, \quad (1.2)$$

where $d\sigma_z$ is the Euclidean area measure, then (1.1) is Blaschke-oscillatory.

In the converse direction, we obtain

Theorem 1.1. *If $A(z)$ is an analytic function in D such that (1.1) is Blaschke-oscillatory, then*

$$\int_D |A(z)|^\alpha d\sigma_z < \infty \quad (1.3)$$

holds for every $\alpha \in (0, \frac{1}{2})$.

For the sharpness of Theorem 1.1, we note that there exists Blaschke-oscillatory equations of the form (1.1) in which the coefficient function $A(z)$ does not satisfy (1.2). Indeed, the functions $f_1(z) = (1 - z) \exp(\frac{1+z}{1-z})$ and $f_2(z) = (1 - z) \exp(-\frac{1+z}{1-z})$, where $z \in D$, are linearly independent solutions of

$$f'' - \frac{4}{(1 - z)^4} f = 0. \quad (1.4)$$

Since f_1 has bounded characteristic, and since $f_2 \in H^\infty$, all solutions of (1.4) have bounded characteristic, and hence (1.4) is Blaschke-oscillatory. This example is a special case of [8, Example 3.3].

Theorems A and 1.1 together extend the following classical result due to Z. Nehari, originally stated in terms of univalent functions, see [10].

Theorem B. *If $A(z)$ is analytic in D satisfying*

$$|A(z)| \leq \frac{a}{(1 - |z|^2)^2}, \quad z \in D, \quad (1.5)$$

where $a = 1$, then (1.1) is disconjugate. Conversely, if $A(z)$ is analytic in D such that (1.1) is disconjugate, then (1.5) holds for $a = 3$.

The remaining part of the present section contains some standard terminology, which will be explained in detail in Section 2.

When studying Problem 2, one of the goals in [5] is to find such a function $A(z)$ having growth as close as possible to (1.2). The method we use requires a further restriction on the prescribed zero sequence $\{z_n\}$ —it has to be uniformly separated. We next state one of such results, which is actually just a special case of [5, Theorem 4.1].

Theorem C. *Let $\{z_n\}$ be an infinite uniformly separated Blaschke sequence of non-zero points in D . Then there is an analytic function $A(z)$ in D such that (1.3) holds for every $\alpha \in (0, \frac{1}{2})$, and that (1.1) possesses a solution having zeros precisely at the points z_n .*

Note that it is just a technical convenience to always assume that all of the points in the prescribed zero sequence $\{z_n\}$ are non-zero. Also note that Theorem C does not say anything about the zeros of any other solution of (1.1), whereas in Theorem 1.1 we assume that all the zero sequences are Blaschke sequences.

To achieve the case $\alpha = \frac{1}{2}$ in Theorem C, the prescribed zero sequence has to be restricted further. Such restriction is

$$S_\beta = \sum_{n=1}^{\infty} (1 - |z_n|)^\beta < \infty, \quad (1.6)$$

where $\beta \in (0, 1]$ is a suitably small fixed constant. In this direction, we next state a special case of [5, Theorem 4.2].

Theorem D. *Let $\{z_n\}$ be an infinite uniformly separated Blaschke sequence of non-zero points in D satisfying (1.6) for some $\beta \in (0, \frac{1}{3})$. Then there is an analytic function $A(z)$ in D , satisfying (1.2), such that (1.1) possesses a solution having zeros precisely at the points z_n .*

Observe that, as the coefficient function $A(z)$ in Theorem D satisfies (1.2), Eq. (1.1) becomes Blaschke-oscillatory.

Next, we generalize Problem 2 for two prescribed zero sequences.

Problem 3. If $\{a_n\}$ and $\{b_n\}$ are two given Blaschke sequences of distinct points in D with no points in common, then find a function $A(z)$, analytic in D , such that (1.1) possesses two linearly independent solutions f_1 and f_2 having zeros precisely at the points a_n and b_n , respectively.

The main goal of the present paper is to solve Problem 3, in the spirit of Theorems C and D, by finding the required function $A(z)$ having minimal growth. Naturally, we need to require that the prescribed zero sequences are uniformly separated. Along with (1.6), another useful restriction for the given zero sequences is

$$\sum_{n=1}^{\infty} (1 - |z_n|)^{\beta} \log \frac{1}{1 - |z_n|} < \infty \quad (1.7)$$

for some $\beta \in (0, 1]$. In Section 2 we will state a few results on sequences satisfying conditions of the form (1.6) and (1.7). Unfortunately, our method in the case of two zero sequences does not give as good growth estimate for $A(z)$ as (1.2), see Theorem 1.2 below.

For technical convenience, we define a sequence $\{z_n\}$ by setting

$$z_{2n-1} = a_n \quad \text{and} \quad z_{2n} = b_n, \quad (1.8)$$

where $\{a_n\}$ and $\{b_n\}$ are given infinite sequences. We make the following two observations:

- (a) $\{z_n\}$ is a Blaschke sequence if and only if the sequences $\{a_n\}$ and $\{b_n\}$ are both Blaschke sequences.
- (b) If $\{z_n\}$ is uniformly separated, then $\{a_n\}$ and $\{b_n\}$ are both uniformly separated. (The converse is obviously not true in general.) Moreover, $\{a_n\}$ and $\{b_n\}$ are sequences of distinct points with no points in common.

Next, we state our main result in which F denotes the class of non-admissible meromorphic functions, and N is the Nevanlinna class, see Section 2.

Theorem 1.2. *Let $\{a_n\}$ and $\{b_n\}$ be two given infinite Blaschke sequences in D such that the sequence $\{z_n\}$ defined in (1.8) is uniformly separated.*

- (a) *If $\{z_n\}$ satisfies (1.7) for $\beta = 1$, then we may construct a function $A(z)$, solving Problem 3, such that $A \in F$ and $E = f_1 f_2 \in N$.*
- (b) *If $\{z_n\}$ satisfies (1.6) for some $\beta \in (0, \frac{1}{3})$, then we may construct a function $A(z)$, solving Problem 3, such that $A \in N$ and $E = f_1 f_2 \in N$. Further, $E', E'' \in N$.*

Note that, although the zero sequences of the solutions f_1 and f_2 of (1.1) in Theorem 1.2 are Blaschke sequences, Eq. (1.1) might not be Blaschke-oscillatory. Indeed, there might exist constants $C_1, C_2 \in \mathbb{C} \setminus \{0\}$ such that the solution $f = C_1 f_1 + C_2 f_2$ of (1.1) has a zero sequence which is not a Blaschke sequence. See [5, Example 3.6] for the case where neither of f_1 and f_2 have no zeros, yet the zeros of $f = f_1 - f_2$ do not form a Blaschke sequence. Hence, Theorem 1.1 cannot be used to obtain a better growth estimate for the coefficient function $A(z)$ in Theorem 1.2.

We also note that natural analogues of Problems 2 and 3 have been studied earlier in the complex plane case, see [9] for a historical review to these studies.

This paper is organized as follows. In Section 2 we recall some standard terminology, and state some auxiliary results related to Blaschke products and uniformly separated sequences. In Section 3 we prove Theorem 1.1. Two lemmas needed to prove the main result are presented in Section 4. The method to prove the main result is introduced in Section 5, while the actual proof can be found in Section 6.

2. Preliminaries

The Nevanlinna class N consists of all functions f , meromorphic in D , having bounded characteristic $T(r, f)$. Further, a meromorphic function f in D is called non-admissible, provided that

$$\limsup_{r \rightarrow 1^-} \frac{T(r, f)}{-\log(1-r)} < \infty.$$

In the latter case, we denote $f \in F$. Clearly, $N \subset F$.

For $0 < p < \infty$, let

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}, \quad 0 < p < \infty,$$

denote the standard L^p -means of the restriction of f to the circle of radius r centered at the origin, and let

$$M_\infty(r, f) = M(r, f) = \max_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|.$$

Then, for $0 < p \leq \infty$, the Hardy space H^p consists of those functions f , analytic in D , for which

$$\|f\|_p = \sup_{0 \leq r < 1} M_p(r, f) < \infty.$$

Clearly, $H^\infty \subset N$. Further, since $p \log x \leq x^p$ for every $0 < p < \infty$ and $x \geq 0$, we conclude the well-known inclusions: $H^p \subset N$ for $0 < p \leq \infty$.

Let $\{z_n\}$ be a sequence of non-zero points in D satisfying (1.6) for $\beta = 1$. Then $\{z_n\}$ is called a Blaschke sequence, and the product

$$B(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}, \quad (2.1)$$

known as the Blaschke product, represents an analytic function in D and has zeros precisely at the points z_n . From now on, $B(z)$ will always denote a Blaschke product of the form (2.1).

The next two auxiliary results related to Blaschke products and Blaschke sequences are restatements of [4, Theorem 1] and [6, Lemma 2.1], respectively.

Theorem E. Let $k \in \mathbb{N}$, and let $\{z_n\}$ be a Blaschke sequence such that (1.6) holds for some $\beta \in (0, \frac{1}{k+1})$. Then, if $m = \frac{1-\beta}{k}$, there is a constant $C(\beta, k) > 0$ such that

$$\int_0^{2\pi} \left| \frac{B^{(k)}(re^{i\theta})}{B(re^{i\theta})} \right|^m d\theta \leq C(\beta, k) S_\beta, \quad \frac{1}{2} < r < 1.$$

Especially, $B^{(k)} \in H^p$ for each $p \in (0, m]$.

Lemma F. Suppose that (1.6) holds for a sequence $\{z_n\}$ of non-zero points in D . For each $r \in [0, 1)$, let $n(r)$ be the number of points z_n lying in the disc $\{z: |z| \leq r\}$. Then

$$n(r) \leq \frac{S_\beta}{(1-r)^\beta}.$$

Next, we point out two results related to the condition (1.7): W. Rudin [13, Theorem I] has showed that if the zeros $\{z_n\}$ of a Blaschke product B satisfy (1.7) for $\beta = 1$, then

$$\int_0^{2\pi} \int_0^1 |B'(re^{i\theta})| dr d\theta < \infty.$$

On the other hand, D. Protas [12, Theorem 3] proved that if (1.7) holds for $\beta = \frac{1}{2}$, then $B' \in H^{\frac{1}{2}}$.

A sequence $\{z_n\}$ of points in D is called separated, if

$$\inf_{n \neq k} \left| \frac{z_n - z_k}{1 - \bar{z}_n z_k} \right| > 0, \quad (2.2)$$

and uniformly separated, if

$$\delta := \inf_k \prod_{n \neq k} \left| \frac{z_n - z_k}{1 - \bar{z}_n z_k} \right| > 0. \quad (2.3)$$

Evidently, a uniformly separated sequence is automatically a Blaschke sequence. Moreover, suppose that $\{z_n\}$ is a sequence of points contained in a Stolz angle in D , hence having a unique accumulation point on ∂D at the vertex of the angle. Then, by [15, Theorem 1], $\{z_n\}$ is uniformly separated if and only if $\{z_n\}$ is separated.

Perhaps the simplest method of constructing uniformly separated sequences is via next result, see [3, Theorem 9.2].

Theorem G. If there is a constant $c \in (0, 1)$ such that

$$1 - |z_{n+1}| \leq c(1 - |z_n|), \quad n = 1, 2, \dots, \quad (2.4)$$

then $\{z_n\}$ is uniformly separated. The condition (2.4) is also necessary if $0 \leq z_1 < z_2 < \dots$.

As an example, suppose that $\{z_n\}$ is a sequence of points in D such that (2.4) holds. Then

$$1 - |z_n| \leq c^{n-1}(1 - |z_1|), \quad n = 2, 3, \dots$$

In particular, $\{z_n\}$ is uniformly separated by Theorem G, and (1.6) holds for any fixed $\beta \in (0, 1]$, because $c^\beta < 1$.

In what follows, $C > 0$ stands for a constant which is independent of all summation indices, but may not be the same at each occurrence.

3. Proof of Theorem 1.1

Let $\{f_1, f_2\}$ be a fundamental system of solutions of (1.1), and define the functions

$$f = f_1 - f_2 \quad \text{and} \quad G = \frac{f_1}{f_2}.$$

Since (1.1) is assumed to be Blaschke-oscillatory, the (possible) zeros of f and the zeros and poles of G are all Blaschke sequences. We also observe that the 1-points of G are the zeros of f , hence they form a Blaschke sequence as well.

Now, by Nevanlinna's second fundamental theorem,

$$T(r, G) \leq N(r, G) + N\left(r, \frac{1}{G}\right) + N\left(r, \frac{1}{G-1}\right) + S(r, G),$$

where the error term $S(r, G)$ satisfies

$$S(r, G) = O\left(\log \frac{1}{1-r}\right) + o(T(r, G)).$$

Applying Lemma F, with $\beta = 1$, we conclude that G is non-admissible.

It is a well-known fact that $2A(z) = S_G(z)$, where

$$S_G = \frac{G'''}{G'} - \frac{3}{2} \left(\frac{G''}{G'} \right)^2$$

is the Schwarzian derivative of G . Next, fix $\alpha \in (0, \frac{1}{2})$, and apply [3, p. 53] and [7, Lemma 3.1] to get

$$\begin{aligned} \int_0^{2\pi} |A(re^{i\theta})|^\alpha d\theta &= O\left(\int_0^{2\pi} \left|\frac{G'''(re^{i\theta})}{G'(re^{i\theta})}\right|^\alpha d\theta + \int_0^{2\pi} \left|\frac{G''(re^{i\theta})}{G'(re^{i\theta})}\right|^{2\alpha} d\theta\right) \\ &= O\left(\left(\frac{1}{1-r} \log \frac{1}{1-r}\right)^{2\alpha}\right). \end{aligned} \quad (3.1)$$

Finally, the assertion (1.3) follows from (3.1), since $\alpha < \frac{1}{2}$.

4. Two lemmas

We need two auxiliary results to prove Theorem 1.2.

Lemma 4.1. Suppose that $\{z_n\}$ is an infinite uniformly separated Blaschke sequence of non-zero points, and let $B(z)$ be the associated Blaschke product. Then

$$|\log B'(z_n)| \leq \log \frac{1}{1 - |z_n|^2} + \pi, \quad n = 1, 2, \dots, \quad (4.1)$$

holds for the principal branch of the logarithm.

Proof. We write $\log B'(z_n) = \log |B'(z_n)| + i \arg B'(z_n)$, where the argument is always in $[-\pi, \pi)$, and observe that

$$|\log B'(z_n)| \leq |\log |B'(z_n)|| + \pi. \quad (4.2)$$

By a direct computation, we have

$$B'(z_n) = \frac{-|z_n|}{z_n(1 - |z_n|^2)} \prod_{k \neq n} \frac{|z_k|}{z_k} \frac{z_k - z_n}{1 - \bar{z}_k z_n}.$$

It is clear that

$$\frac{\delta}{1 - |z_n|^2} \leq |B'(z_n)| \leq \frac{1}{1 - |z_n|^2}, \quad (4.3)$$

where $\delta > 0$ is the constant in (2.3). Now, (4.2) and (4.3) yield the assertion. \square

Lemma 4.2. Suppose that $\{z_n\}$ is an infinite uniformly separated Blaschke sequence of non-zero points satisfying

$$\sum_{n=1}^{\infty} (1 - |z_n|) \log \frac{1}{1 - |z_n|} < \infty. \quad (4.4)$$

Let $\{\sigma_n\}$ be a sequence of complex points, not necessarily distinct, such that

$$|\sigma_n| = O\left(\log \frac{1}{1 - |z_n|}\right). \quad (4.5)$$

Then there exists a function $g \in H^1$ such that $g(z_n) = \sigma_n$ for every n . Moreover, if (4.4) is replaced by

$$\sum_{n=1}^{\infty} (1 - |z_n|)^{\beta} < \infty \quad (4.6)$$

for some $\beta < \frac{1}{3}$, then this same $g \in H^1$ satisfies $g' \in H^{\frac{1-\beta}{2+\beta}}$ and $g'' \in H^{\frac{1-\beta}{3}}$.

Proof. Throughout the proof, we denote $\arg(z) = \theta$. For each $k \in \mathbb{N}$, we define the auxiliary functions

$$c_k(z) = \prod_{m \neq k} \frac{|z_m|}{z_m} \frac{z_m - z}{1 - \bar{z}_m z} = \frac{z_k}{|z_k|} \frac{1 - \bar{z}_k z}{z_k - z} B(z),$$

where $B(z)$ is the Blaschke product associated with $\{z_n\}$. We claim that the series

$$g(z) = \sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)^2}{(1 - \bar{z}_k z)^2} \sigma_k \frac{c_k(z)}{c_k(z_k)} \quad (4.7)$$

is the function that we are looking for. We note that $g(z)$ is the series $f(z)$ in [3, p. 153], with $p = 1$ and $w_k = (1 - |z_k|^2)\sigma_k$.

Suppose first that (4.4) holds. We have $|c_k(z_k)| \geq \delta > 0$ by (2.3). Therefore, by (4.4) and (4.5), we get

$$|g(z)| = O\left(\sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)^2}{|1 - \bar{z}_k z|^2} \log \frac{1}{1 - |z_k|}\right) = O\left(\frac{1}{1 - |z|}\right),$$

which shows that $g(z)$ converges uniformly in compact subsets of D , and hence is analytic in D . Moreover, we observe that $g(z_n) = \sigma_n$ for every n , and

$$\begin{aligned} \int_0^{2\pi} |g(z)| d\theta &= O\left(\sum_{k=1}^{\infty} (1 - |z_k|^2)^2 \log \frac{1}{1 - |z_k|} \int_0^{2\pi} \frac{d\theta}{|1 - \bar{z}_k z|^2}\right) \\ &= O\left(\sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)^2}{1 - |z_k z|^2} \log \frac{1}{1 - |z_k|}\right) = O(1), \end{aligned}$$

so that $g \in H^1$.

Suppose then that (4.6) holds. We will make use of the simple fact that then

$$\sum_{n=1}^{\infty} (1 - |z_n|)^{\frac{1}{3}} \log \frac{1}{1 - |z_n|} < \infty \quad (4.8)$$

holds as well. It suffices to prove that $g'' \in H^{\frac{1-\beta}{3}}$, where $g(z)$ is the function in (4.7), for this will imply $g' \in H^{\frac{1-\beta}{2+\beta}}$ by [3, Theorem 5.12].

Differentiating g , we get $g'' = g_1 + g_2 + g_3$, where

$$\begin{aligned} g_1(z) &= \sum_{k=1}^{\infty} (1 - |z_k|^2)^2 \frac{\sigma_k}{c_k(z_k)} \frac{c_k''(z)}{(1 - \bar{z}_k z)^2}, \\ g_2(z) &= \sum_{k=1}^{\infty} (1 - |z_k|^2)^2 \frac{\sigma_k}{c_k(z_k)} \frac{4\bar{z}_k c_k'(z)}{(1 - \bar{z}_k z)^3} \end{aligned}$$

and

$$g_3(z) = \sum_{k=1}^{\infty} (1 - |z_k|^2)^2 \frac{\sigma_k}{c_k(z_k)} \frac{6\bar{z}_k^2 c_k(z)}{(1 - \bar{z}_k z)^4}.$$

To show that $g_1(z)$ is an analytic function in D , we first observe that Cauchy's formula gives

$$|c_k''(z)| = O\left(\left(\frac{1}{1 - |z|}\right)^2\right)$$

uniformly for each k . Therefore, by (4.5) and (4.8),

$$|g_1(z)| = O\left(\frac{1}{(1-|z|)^2} \sum_{k=1}^{\infty} \frac{(1-|z_k|^2)^2}{|1-\bar{z}_k z|^2} \log \frac{1}{1-|z_k|}\right) = O\left(\left(\frac{1}{1-|z|}\right)^{\frac{11}{3}}\right),$$

so that $g_1(z)$ converges uniformly in compact subsets of D , and hence is analytic in D . Similarly, we may show that $g_2(z)$ and $g_3(z)$ are analytic in D .

We proceed to show that $g_1 \in H^{\frac{1-\beta}{3}}$, $g_2 \in H^{\frac{2(1-\beta)}{3(2-\beta)}}$, and $g_3 \in H^{\frac{1}{3}}$. Then, as $\frac{1-\beta}{3} \leq \min\left\{\frac{2(1-\beta)}{3(2-\beta)}, \frac{1}{3}\right\}$, we may conclude that $g'' \in H^{\frac{1-\beta}{3}}$, which will complete the proof.

Proof of $g_3 \in H^{\frac{1}{3}}$. By Lemma in [3, p. 57], and by (4.8), we have

$$\begin{aligned} \int_0^{2\pi} |g_3(z)|^{\frac{1}{3}} d\theta &= O\left(\sum_{k=1}^{\infty} (1-|z_k|)^{\frac{2}{3}} |\sigma_k|^{\frac{1}{3}} \int_0^{2\pi} \frac{d\theta}{|1-\bar{z}_k z|^{\frac{4}{3}}}\right) \\ &= O\left(\sum_{k=1}^{\infty} (1-|z_k|)^{\frac{1}{3}} \left(\log \frac{1}{1-|z_k|}\right)^{\frac{1}{3}}\right) = O(1), \end{aligned}$$

and we are done.

Proof of $g_2 \in H^{\frac{2(1-\beta)}{3(2-\beta)}}$. First, we show that

$$\|c'_k\|_{1-\beta} \leq C \quad (4.9)$$

holds uniformly for every k . Let $b_k(z) = \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \bar{z}_k z}$ denote the k th Blaschke factor. By the definition, we have $b_k(z)c_k(z) = B(z)$, so that

$$b_k(z)c'_k(z) = B'(z) - b'_k(z)c_k(z).$$

A computation based on the Hölder inequality and the Poisson kernel gives us $\|b'_k c_k\|_{1-\beta} \leq C$ uniformly for each k . Also, by Theorem E, we have $B' \in H^{1-\beta}$. Therefore, $\|b_k c'_k\|_{1-\beta} \leq C$ uniformly for each k , which proves (4.9).

Second, we denote $p_2 = \frac{2(1-\beta)}{3(2-\beta)}$, and make use of Hölder's inequality, with the conjugate indices

$$u_2 = \frac{1-\beta}{p_2} = \frac{6-3\beta}{2} \quad \text{and} \quad v_2 = \frac{1-\beta}{1-\beta-p_2} = \frac{6-3\beta}{4-3\beta},$$

together with (4.8) and (4.9) to conclude that

$$\begin{aligned} \int_0^{2\pi} |g_2(z)|^{p_2} d\theta &= O\left(\sum_{k=1}^{\infty} (1-|z_k|)^{2p_2} |\sigma_k|^{p_2} \int_0^{2\pi} \frac{|c'_k(z)|^{p_2}}{|1-\bar{z}_k z|^{3p_2}} d\theta\right) \\ &= O\left(\sum_{k=1}^{\infty} (1-|z_k|)^{2p_2} |\sigma_k|^{p_2} \left(\int_0^{2\pi} \frac{d\theta}{|1-\bar{z}_k z|^{3p_2 v_2}}\right)^{\frac{1}{v_2}}\right) \end{aligned}$$

$$= O\left(\sum_{k=1}^{\infty} (1 - |z_k|)^{2p_2 - \frac{3p_2 v_2 - 1}{v_2}} \log \frac{1}{1 - |z_k|}\right) = O(1),$$

and we are done.

Proof of $g_1 \in H^{\frac{1-\beta}{3}}$. First, we note that an argument similar to the one used in proving (4.9) shows that

$$\|c_k''\|_{\frac{1-\beta}{2}} \leq C \quad (4.10)$$

holds uniformly for each k . Then, denoting $p_1 = \frac{1-\beta}{3}$ and applying Hölder's inequality, with the conjugate indices

$$u_1 = \frac{1-\beta}{2p_1} = \frac{3}{2} \quad \text{and} \quad v_1 = \frac{1-\beta}{1-\beta-2p_1} = 3,$$

we get the assertion by modifying the corresponding proof for the function g_2 . We just point out that

$$2p_1 v_1 = 2(1-\beta) > 1 \quad \text{and} \quad 2p_1 - \frac{2p_1 v_1 - 1}{v_1} = \frac{1}{v_1} = \frac{1}{3}.$$

This completes the proof. \square

5. Bank–Laine functions

The paper [1] by S. Bank and I. Laine has become one of the cornerstones in the zero distribution theory of complex differential equations. In this section we state some of the main ideas of [1] in the unit disc setting, and we will then use these ideas to prove Theorem 1.2 in Section 6.

A function E , analytic in D , is called a Bank–Laine function, if at every zero ζ of E it is true that either $E'(\zeta) = 1$ or $E'(\zeta) = -1$. Bank–Laine functions are closely related to (1.1) through the non-linear differential equation

$$-4A(z)E^2 = 1 - (E')^2 + 2EE'' \quad (5.1)$$

(originally developed in [1]) and the following lemma, see [2, Lemma C] and [14, Lemmas 1.1 and 1.2].

Lemma H. *If E is a Bank–Laine function, then $A(z)$, defined by (5.1), is analytic in D , and E is a product of two linearly independent solutions f_1 and f_2 of (1.1), normalized so that the Wronskian $W(f_1, f_2) = 1$.*

Conversely, if E is a product of two linearly independent solutions of (1.1), where $A(z)$ is defined by (5.1), then E is a Bank–Laine function.

We now give a general approach to solve Problem 3 by using Bank–Laine functions. To this end, let $B(z)$ be a Blaschke product such that its only zeros are $\{z_n\}$, see (1.8).

Interpolation problem. Find an analytic function g in D such that

$$e^{g(z)} = \begin{cases} -\frac{1}{B'(a_n)}, & \text{if } z = a_n, \\ \frac{1}{B'(b_n)}, & \text{if } z = b_n. \end{cases}$$

Suppose that g solves the interpolation problem above. Then the function $E = Be^g$ satisfies $E' = B'e^g + Bg'e^g$, and hence is a Bank–Laine function:

$$E(z) = \begin{cases} -1, & \text{if } z = a_n, \\ 1, & \text{if } z = b_n. \end{cases} \quad (5.2)$$

By the reasoning in [14], based on Lemma H, $E = f_1 f_2$, where f_1 and f_2 are solutions of (1.1) having zeros precisely at the points a_n and b_n , respectively.

To solve the interpolation problem above, we need to find an analytic function g in D such that

$$g(z) = \begin{cases} \log\left(-\frac{1}{B'(a_n)}\right), & \text{if } z = a_n, \\ \log\left(\frac{1}{B'(b_n)}\right), & \text{if } z = b_n. \end{cases} \quad (5.3)$$

Observe that any branch of the logarithm can be chosen due to the periodic nature of the exponential function.

So, to prove Theorem 1.2, we construct a function g satisfying (5.3) by using the lemmas from Section 4. Then we estimate the growth of the function E , and finally we estimate the growth of the coefficient function $A(z)$ by means of (5.1). For example, we have

$$\begin{aligned} T(r, A) &\leq 2T(r, E) + 2T\left(r, \frac{E'}{E}\right) + T\left(r, \frac{E''}{E}\right) + O(1) \\ &\leq 2T(r, E) + 2m\left(r, \frac{E'}{E}\right) + m\left(r, \frac{E''}{E}\right) + 3N\left(r, \frac{1}{E}\right) + O(1). \end{aligned} \quad (5.4)$$

The details are presented in Section 6.

6. Proof of Theorem 1.2

The proof relies on the method described in Section 5. We define a sequence $\{\sigma_n\}$ of complex points by setting

$$\sigma_n = \log(-1)^n - \log B'(z_n) = \log\left(\frac{(-1)^n}{B'(z_n)}\right), \quad n = 1, 2, \dots,$$

where the branches of the logarithms on the left are principal branches for every n , and the branch for the logarithm on the right will be fixed accordingly. Since $\{z_n\}$ is uniformly separated, we have

$$|\sigma_n| = O\left(\log \frac{1}{1 - |z_n|}\right) \quad (6.1)$$

uniformly for each n by Lemma 4.1. The further assumptions on $\{z_n\}$ in parts (a) and (b) are (4.4) and (4.6), respectively. Hence, by Lemma 4.2, we can construct a function $g \in H^1$

such that $g(z_n) = \sigma_n$. Such a function g will automatically satisfy (5.3) by the definition of the sequence $\{z_n\}$, see (1.8).

Conclusion of (a). Suppose that (4.4) holds. Since $B \in H^\infty$ and $e^g \in N$, we clearly have $E \in N$. By (5.4) and the standard logarithmic derivative estimates, we get

$$T(r, A) = O\left(\log \frac{1}{1-r}\right),$$

which shows that $A(z)$ is non-admissible. By Lemma H, it is also clear that $A(z)$ is analytic in D .

Conclusion of (b). Suppose that (4.6) holds. Differentiating $E = Be^g \in N$, we obtain

$$\frac{E'}{E} = \frac{B'}{B} + g' \quad \text{and} \quad \frac{E''}{E} = \frac{B''}{B} + 2\frac{B'}{B}g' + g'' + (g')^2.$$

The functions B' and B'' belong to certain Hardy spaces by Theorem E, hence $\frac{B'}{B}$ and $\frac{B''}{B}$ are both in N . On the other hand, the functions g' and g'' belong to certain Hardy spaces as well by Lemma 4.2, so that $\frac{E'}{E}$ and $\frac{E''}{E}$ are both in N . We now conclude $A \in N$ via (5.1). By Lemma H, it is also clear that $A(z)$ is analytic in D .

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